Problem 1.27

Prove that the divergence of a curl is always zero. *Check* it for function $\mathbf{v}_a$ in Prob. 1.15.

Solution

Evaluate the divergence of a curl explicitly.

$$\begin{aligned}
\nabla \cdot (\nabla \times \mathbf{A}) &= \left(\sum_{i=1}^{3} \boldsymbol{\delta}_i \frac{\partial}{\partial x_i}\right) \cdot \left[\left(\sum_{j=1}^{3} \boldsymbol{\delta}_j \frac{\partial}{\partial x_j}\right) \times \left(\sum_{k=1}^{3} \boldsymbol{\delta}_k A_k\right)\right] \\
&= \left(\sum_{i=1}^{3} \boldsymbol{\delta}_i \frac{\partial}{\partial x_i}\right) \cdot \left[\sum_{j=1}^{3}\sum_{k=1}^{3} (\boldsymbol{\delta}_j \times \boldsymbol{\delta}_k) \frac{\partial A_k}{\partial x_j}\right] \\
&= \left(\sum_{i=1}^{3} \boldsymbol{\delta}_i \frac{\partial}{\partial x_i}\right) \cdot \left(\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3} \boldsymbol{\delta}_l \varepsilon_{jkl} \frac{\partial A_k}{\partial x_j}\right) \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3} (\boldsymbol{\delta}_i \cdot \boldsymbol{\delta}_l) \varepsilon_{jkl} \frac{\partial}{\partial x_i} \frac{\partial A_k}{\partial x_j} \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3} \delta_{il} \varepsilon_{jkl} \frac{\partial}{\partial x_i} \frac{\partial A_k}{\partial x_j} \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} \varepsilon_{jki} \frac{\partial}{\partial x_i} \frac{\partial A_k}{\partial x_j} \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} \varepsilon_{ijk} \frac{\partial^2 A_k}{\partial x_i \partial x_j} \\
&= \sum_{j=1}^{3}\sum_{i=1}^{3}\sum_{k=1}^{3} \varepsilon_{jik} \frac{\partial^2 A_k}{\partial x_j \partial x_i} \qquad \text{(Let } i \text{ be } j \text{ and let } j \text{ be } i\text{.)} \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} \varepsilon_{jik} \frac{\partial^2 A_k}{\partial x_j \partial x_i} \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} \varepsilon_{jik} \frac{\partial^2 A_k}{\partial x_i \partial x_j} \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} (-\varepsilon_{ijk}) \frac{\partial^2 A_k}{\partial x_i \partial x_j} \\
&= -\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} \varepsilon_{ijk} \frac{\partial^2 A_k}{\partial x_i \partial x_j} \\
&= 0
\end{aligned}$$

Then verify it for $\mathbf{v}_a = x^2\hat{\mathbf{x}} + 3xz^2\hat{\mathbf{y}} - 2xz\hat{\mathbf{z}}$.

$$\begin{aligned}
\nabla \cdot (\nabla \times \mathbf{v}_a) &= \nabla \cdot \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & 3xz^2 & -2xz \end{vmatrix} \\
&= \nabla \cdot \left\{ \hat{\mathbf{x}} \left[\frac{\partial}{\partial y}(-2xz) - \frac{\partial}{\partial z}(3xz^2) \right] - \hat{\mathbf{y}} \left[\frac{\partial}{\partial x}(-2xz) - \frac{\partial}{\partial z}(x^2) \right] + \hat{\mathbf{z}} \left[\frac{\partial}{\partial x}(3xz^2) - \frac{\partial}{\partial y}(x^2) \right] \right\} \\
&= \nabla \cdot \left\{ \hat{\mathbf{x}} \left[(0) - (6xz) \right] - \hat{\mathbf{y}} \left[(-2z) - (0) \right] + \hat{\mathbf{z}} \left[(3z^2) - (0) \right] \right\} \\
&= \nabla \cdot (-6xz\hat{\mathbf{x}} + 2z\hat{\mathbf{y}} + 3z^2\hat{\mathbf{z}}) \\
&= \frac{\partial}{\partial x}(-6xz) + \frac{\partial}{\partial y}(2z) + \frac{\partial}{\partial z}(3z^2) \\
&= (-6z) + (0) + (6z) \\
&= 0
\end{aligned}$$